THE LIMIT NOTATION AS A MEDIATOR OF MATHEMATICAL DISCOURSE

Beste Güçler
University of Massachusetts Dartmouth
bgucler@umassd.edu

Symbols play important roles in higher-level mathematical thinking by providing flexibility and reducing cognitive lo ad. However, they often have a dual nature since they can signify both processes and products of mathematics. The limit notation is considered to be a visual and symbolic mediator that reflects such duality, which presents challenges for students. This study focuses on one instructor's utilization of the limit notation and examines how it mediates his discourse on limits. The findings indicate that the process and product aspects of limit were both present in the instructor's use of the limit notation. Although he differentiated between these two aspects depending on the mathematical context, the distinction remained implicit for the students in the classroom. The study concludes that it is important for teachers to unpack the meanings inherent in symbols in the classroom to enhance mathematical communication.

Keywords: Classroom Discourse, Post-Secondary Education, Advanced Mathematical Thinking

Introduction

Mathematics is often viewed as an embodied activity where learners generate mathematical meaning through their perceptions, senses, and experiences in the real world (Lakoff & Núñez, 2000). Although initial learning of mathematical objects may be based on learners' sensual perceptions of physical objects, such an approach can hinder learning of higher-level mathematical skills (Gray & Tall, 2001). Indeed, a significant characteristic of mathematical sophistication is generalizing context-bound thinking to context-free (abstract) thinking.

Symbols play important roles in such generalizations and the transition to higher order mathematical thinking. There is a reflexive relationship between symbolizing and mathematical meaning (Cobb, 2000) indicating that the meanings learners attribute to the signs are as critical as the syntactical aspects of symbolism. Radford and Puig (2007) argue that signs are deposited with historical cognitive activity and the social practices they mediate are not transparent to the students. According to them, learners' awareness of the historical intelligence embedded in symbols requires their active participation in a sense-making process.

Learners' struggles with the sense-making process, as they interact with symbols, may be due to the ambiguous nature of symbols. Cobb (2000) mentions that a concrete mathematical symbol "can serve different symbolizing functions and might, in fact, be used non-symbolically" (p. 18). Similarly, Gray and Tall (1994) point to the ambiguity of symbols by highlighting that symbols can signify both processes and products of mathematical activity. They argue that, although the ambiguity of symbols may present students with challenges, it is also an essential element of mathematical thinking since it supports flexible interpretation of mathematical notations by simplifying "the cognitive complexity of process-concept duality by the notational convenience of process-product ambiguity" (Gray & Tall, 1994, p. 121).

Limit is a mathematical concept that reflects this process-product ambiguity. Research shows that the informal and formal aspects of limit are based on different metaphors, supporting different realizations of the concept as a process or a product (e.g., Cornu, 1991; Güçler, 2013; Lakoff & Núñez, 2000; Tall & Vinner, 1981). This ambiguity can also interfere with thinking

about the representational aspects of the concept such as the limit notation (Bagni, 2005; Gray & Tall, 1994; Tall & Vinner, 1981). Gray and Tall (1994) indicate that the issue is not whether learners consider limits as processes or products, but whether they can consider limit as both, depending on the mathematical context. They argue that such a utilization of notation requires that the difference "between process and concept is maintained at all times" in the realization of limits (Gray & Tall, 1994, p. 121).

Teachers' discourse in the classroom can play critical roles in students' thinking about limit given that symbols may play ambiguous roles in mathematics and that the syntax as well as the meaning of symbols may not be transparent to students. Teachers may also help students in differentiating and amalgamating the process and product aspects of limit. However, in order for this to happen, it may not be enough for teachers to only use mathematical language and symbols accurately; they need to also make such elements of their discourse transparent to their students in the classroom (Güçler, 2013). This study uses a discursive approach to address the following question: How does the limit notation mediate one instructor's discourse on limits in a beginning-level undergraduate calculus classroom? Particular sub-questions include: How does the instructor utilize the limit notation? What does the limit notation signify in the instructor's discourse? Are the distinctions between the process and product aspects of limit transparent in the instructor's discourse?

Theoretical Framework

The process-product duality of symbolism and learners' need to form conceptual entities from dynamic processes are addressed through different conceptualizations in the literature such as encapsulation (Dubinsky, 1991; Gray & Tall, 1994, 2001) and reification (Sfard, 1991, 1992). These approaches provide significant insights about the nature of mathematical objects and some of the challenges related to mathematical learning, but do not take into account the socio-cultural nature of generating signs and their meaning. The assumption of this study is that signs and artifacts are cultural products; they support semiotic mediation as their meanings are generated and negotiated by communities engaged in mathematical discourse (Vygotsky, 1978; Radford & Puig, 2007).

Sfard's (2008) more recent approach integrates mathematical cognition with mathematical communication with the aim of dissolving the dichotomy between the individual and social aspects of learning. She formulates thinking as an individualized form of communication and considers learning as change in one's discourse through increased participation in communities of practice. The main unit of analysis in her approach is discourse, which is defined as "different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). This work is based on her discursive approach that formulates mathematics as a particular discourse that is distinguishable by its word use, visual mediators, routines, and endorsed narratives (Sfard, 2008). Word use refers to participants' use of mathematical words in their discourse. Visual mediators refer to all the visible objects that are created and acted on for mathematical communication. Routines refer to the repetitive meta-level rules in the actions of participants as they substantiate their narratives about mathematics. Endorsed narratives refer to the set of utterances the participants consider as true given their word use, visual mediators, and routines.

In the context of her discursive approach, Sfard (2008) extends the notion of reification to objectification. She notes that a critical feature of word use is objectification, which occurs

through reification and alienation. Reification "is the act of replacing sentences about processes and actions with propositions about states and objects" (Sfard, 2008, p. 44), whereas alienation refers to "using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings" (Sfard, 2008, p. 295). Said differently, objectification changes the talk about processes to the talk about end-states or mathematical entities. In this study, the instructor's use of the limit notation as a visual mediator is examined in relation to his word use, routines, and endorsed narratives.

Methodology

This work is part of a case study that investigated one instructor's and his students' discourses on limits in a beginning-level undergraduate calculus classroom at a large mid-western university in the U.S. The data sources for the larger study included eight video-taped classroom sessions during which the instructor talked about limits and continuity; a survey given to all the students at the end of their classroom discussions on limits; and four students' responses to task-based interview sessions on limits. For the purposes of this paper, the data consists of the instructor's classroom sessions (see Güçler (2013) for a detailed analysis of the entire study). The classroom observations were transcribed both with respect to the instructor's utterances and actions. The transcripts included the snapshots of everything the instructor wrote and drew on the board.

For the analysis of the visual mediators, an inventory of all the visual mediators the instructor used was collated from the transcripts. They were then compiled into three categories: graphs, written words, and symbolic representation. The focus of this study is on one visual mediator, namely the limit notation as a symbolic mediator of mathematical discourse. Note that the description of what type of mediator used gives little information about mathematical discourse, so the discussion of visual mediators needs to also include how and when they were used, what kinds of narratives about limit the notation supported, and how the instructor talked about the notation. Therefore, the analysis of visual mediators is presented with the analyses of the instructor's word use, routines, and endorsed narratives in the context of the limit notation.

The instructor's word use was analyzed in terms of the degree of objectification. When talking about the limit notation, his word use was considered operational if he referred to limit as a dynamic process, and objectified if he referred to limit as a distinct mathematical object—a static number that is obtained at the end of the limit process. For the analysis of routines, the focus was on the instructor's actions to elicit when and how he used the limit notation (Sfard, 2008). Finally, for the analysis of the endorsed narratives, the focus was on two meta-level narratives that were substantiated by the instructor through his word use and routines of using the limit notation: limit is a process and limit is a number.

Results

Dr. Brenner (a pseudonym) had 775 utterances about limits throughout the eight lessons. About 82% of those utterances were coded as objectified, indicating that he mainly talked about limit as a static mathematical entity. There were two limit-related contexts in which Dr. Brenner's operational word use took place: the informal definition of limit and computing limits. In these two contexts, he shifted his word use from operational to objectified and vice versa. The analysis in this section examines how he communicated in these contexts with a particular attention to the role of the limit notation in his discourse.

When Dr. Brenner introduced the informal definition of limit, he also introduced the limit notation as shown in Figure 1 and Table 1.

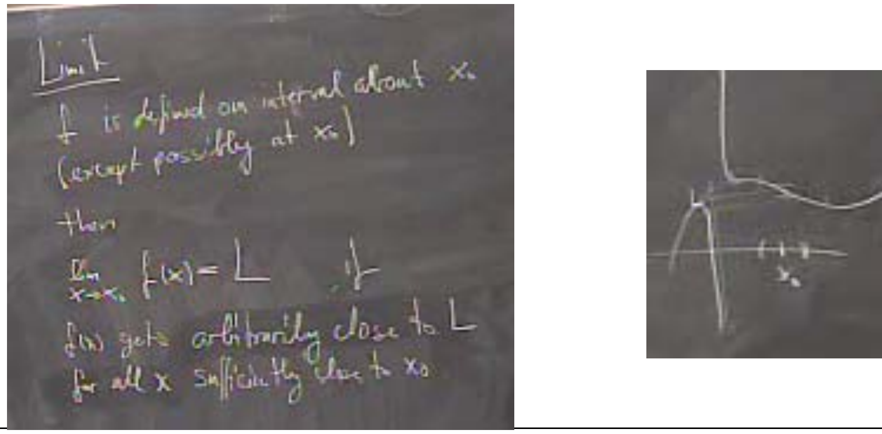

Figure 1: Video Snapshot of Dr. Brenner's Informal Definition of Limit

Table 1: Dr. Brenner's Introduction to the Informal Definition of Limit

What is said	What is done
[1] What is a limit?	He writes "f is defined on an interval about x_0" on the board and starts drawing a graph (See Figure 1).
a) x_0 is here. I have some function and I want to make sure it is defined at least near x_0.	
b) So if I take a small interval here, this function is defined.	He draws an open interval around x_0.
c) So we don't ask it to be defined at x_0 but at least nearby.	He writes "except possibly at x_0" on the board.
[2] Then we say as x approaches x_0 of the function $f(x)$ equals some number L if…(does not finish his sentence)	He writes $\lim\limits_{x \to x_0} f(x)$ when he says these and then pauses.
[3] What does it mean that the limit of the function as x approaches x_0 is L?	He shows the notation and turns back to the graph he drew.
[4] It means that the function value, if I get closer and closer to x_0, should approach some number L.	He puts L on the graph he drew and then completes writing $\lim\limits_{x \to x_0} f(x)$ as being equal to L. (See Figure 1).
[5] I want to say it [$f(x)$] is arbitrarily close to L for all x sufficiently close to x_0.	He writes these on the board (See Figure 1).
[6] So this is what I would want to call the intuitive definition of a limit.	
[7] So we say the limit is L if I can make the function values to be arbitrarily close to L if I choose my values of x sufficiently close to x_0.	

There are a couple observations that can be made from this introduction. Dr. Brenner introduced the limit notation quite tacitly (Table 1, [2], [3]), as if it was an unproblematic mediator in the discourse on limits. When he referred to $\lim_{x \to x_0} f(x)$, he mentioned the x values approaching x_0 (Table 1, [2], [3]), which signified limit as a process through dynamic motion[1]. He paused at this stage and then focused on the behavior of the function of which he drew an example (Figure 1). In other words, the signs $\lim_{x \to x_0} f(x)$ prompted the instructor to examine the behavior of the function, which he did so using words signifying motion, such as getting "closer and closer to" and "approach" (Table 1, [4]), signifying limit as a process. After the examination of the function's behavior, Dr. Brenner completed the right-hand side of the limit notation as being equal to the number L (writing $\lim_{x \to x_0} f(x) = L$) that signifies the number (product) obtained at the end of the limit process. He then shifted his operational word use signifying motion—the process view of limit—to those signifying proximity (Table 1, [5], [7]) consistent with the formal theory of limits—the static and product view of limit—but these utterances did not necessarily cohere with his action of finding the limit. Said differently, although both a process and a product view of limit were present in his discourse when attending to the limit notation, he did not explain these two aspects explicitly in the classroom.

Similar patterns emerged in the instructor's use of the limit notation as a routine when computing limits. Table 2 shows Dr. Brenner's utterances when he computed $\lim_{x \to 2} \frac{1}{x-1}$.

Table 2: Dr. Brenner's Utterances When Computing $\lim_{x \to 2} \frac{1}{x-1}$.

What is said	What is done
[1] What is this limit?	He writes $\lim_{x \to 2} \frac{1}{x-1}$ on the board.
[2] Let's see. If x gets closer and closer to two, this quantity gets closer and closer to one over two minus one.	He shows $x \to 2$ and then shows $\frac{1}{x-1}$. He says these verbally and does not write anything on the board.
[3] It [the function] is very close to one over one.	
[4] So the closer x gets to two, the closer this will get to one.	He shows $\frac{1}{x-1}$.
[5] This limit is one.	He writes $\lim_{x \to 2} \frac{1}{x-1} = 1$; no graph is drawn.

In this computation, Dr. Brenner's initial use of the limit notation prompted him to explore the behavior of the function near the limit point. While doing so, he uttered operational phrases like getting "closer and closer to" and getting to the limit value (Table 2, [2], [4]) and talked about limit as a process. At the end of his investigation, his word use was objectified as he referred to limit as the product (a number) of the process (Table 2, [5]) and wrote the limit being equal to 1 in the limit notation. Note also that his deictic references were about the function values (Table 2, [2-4]) when he attended to the behavior of the function whereas his reference

was about the limit in the end (Table 2, [5]), which was similar to the patterns observed in the context of the informal definition of limit. Dr. Brenner's deictic utterances in Table 1 ([4], [5]) are about the function whereas his final utterance (Table 1, [7]) is about the limit of the function.

These two examples are representative of the instructor's use of the limit notation throughout the 64 limit computation problems on which he worked. In the contexts of the informal definition and computing limits, his routine of using the limit notation can be summarized as follows:

1. The symbol $\lim_{x\to x_0} f(x)$ prompted him to examine the behavior of $f(x)$. When reading $x \to x_0$ in the limit notation, he mentioned x values approaching x_0, consistent with the textbook's introduction of the limit notation.
2. He then examined the behavior of the functions uttering phrases of the form: What does the function values "approach/get closer and closer to/become/go to" as the x values "approach/get closer and closer to/become/go to" x_0 ? (Symbolically represented as $f(x) \to ?$ as $x \to x_0$).
3. At the end of the examination of the behavior of the functions, he focused on the end result of the process and talked about the value L (if it existed) as being *equal* to the limit of the function (Symbolically represented as $\lim_{x\to x_0} f(x) = L$).

The stages of Dr. Brenner's routines and word use suggest that he endorsed the narrative *limit is a process* when he referred to the left-hand side of the limit notation and when he explored the behavior of functions near the limit points. In contrast, he endorsed the narrative *limit is a number*—a product view of limit—when he completed his examination of the behavior of functions and completed the right-hand side of the limit notation.

The routines in the instructor's use of the limit notation can be elicited from his actions and word use, but a critical characteristic of these routines is that they are meta-level, i.e., the metarules forming the routines of discourse are often tacit (Sfard, 2008). That the instructor did not make his discourse on the limit notation an explicit topic of discussion in the classroom is consistent with the tacit nature routines.

Although the students' use of the limit notation is not a main focus of this study due to space constraints, a brief summary is provided to give some information about the role of the notation in their discourse. The analyses of the surveys and the interview sessions showed the following stages in students' routines of using the limit notation:

1. The symbol $\lim_{x\to x_0} f(x)$ prompted the students to examine the behavior of $f(x)$. In most cases, the students did not explicitly talk about $x \to x_0$ in the limit notation. When they did, they mentioned x values approaching/getting closer and closer to x_0 .
2. They then examined the behavior of the functions uttering phrases of the form: What does the function values "approach/get closer and closer to/become/go to" as the x values "approach/get closer and closer to/become/go to" x_0 ? (Symbolically represented as $f(x) \to ?$ as $x \to x_0$).
3. At the end of the examination of the behavior of the functions, the students did not focus on the end result of the process. Instead, their word use remained operational. When asked what the limit of $f(x)$ is, they uttered phrases of the form: "It approaches/gets closer and closer to L" indicating that they did not differentiate the function values approaching L from the limit value being *equal to L*.

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Students' routines as well as word use suggested that, in the context of the limit notation, they only endorsed the narrative *limit is a process* and did not consider *limit as a number*. This finding provides some evidence that, unlike the instructor, the students did not consider the process and product aspect of limit as inherent in the symbolism. During the surveys and the interview sessions, there was no clear evidence in students' discourse indicating their awareness of the different meanings the limit notation supported.

These results do not suggest that the instructor's discourse caused the students' difficulties with the limit notation (for which the study has no evidence). Yet, the findings suggest that there was clearly some miscommunication between Dr. Brenner and his students in terms of the limit notation and the different meanings of limit it supported.

Dr. Brenner's use of the limit notation took into account both the process and product aspect of limit. Further, he kept the two aspects of the notation distinct throughout his discourse, as evidenced by his word use and the different narratives he endorsed. However, accurate use of language and symbol use did not seem to be enough for his students to infer the subtleties of symbol use. Dr. Brenner may have enhanced classroom communication had he spent more time introducing the limit notation and unpacking the meaning of the notation in different contexts related to limits.

Conclusion and Discussion

In Dr. Brenner's discourse, the instances he used operational and objectified utterances about limit were clearly distinguished when referring to the limit notation as a visual mediator of discourse. In other words, the instructor maintained the differences between the process and product aspects of the limit at all times as indicated by Gray and Tall (1994). However, such a distinction may not be present in the students' discourse, especially during the initial stages of their learning.

The ambiguity of process-product duality is present in the limit notation. Such ambiguity has its advantages in mathematical thinking in terms of the flexible use of symbols and reduction of cognitive load when thinking about mathematical objects (Gray & Tall 1994, 2001; Sfard, 1991). On the other hand, researchers also argue that such duality creates many challenges for learners. The findings of the study indicate that it is important not to take the communicative power of symbols, or any visual mediator, for granted. In other words, teachers need to unpack the duality inherent in symbolism and the different mathematical meanings that can be generated from the same symbol.

Experts of mathematics are often aware of the dual nature of symbols. We see in Dr. Brenner's discourse how he differentiated between the process and product aspects of limit depending on the mathematical situation, indicating the flexibility of his use of the limit notation. The routines with which he substantiated the endorsed narratives about limit (*limit is a process* and *limit is a number*), however, remained implicit to the students in his classroom. Making the tacit metarules constituting the routines explicit topics in the classroom can be useful in unpacking the meaning-making and negotiation processes behind mathematical symbolism. In addition, it can help highlight the instances of communication and miscommunication between students and teachers.

Note that, although the focus of this study was on the limit notation, the discussions about the instructor's discourse were meaningful only when the visual mediator was analyzed in conjunction with his word use, routines, and endorsed narratives. Sfard's (2008) framework is useful to show the interrelations among all the elements of mathematical discourse, indicating

that successful participation in the discourse on mathematics requires the orchestration of all these elements. These elements of discourse can also help teachers in terms of what they may need to focus on in order to enhance mathematical communication in their classrooms.

Endnote

[1] The instructor's utterances about the limit notation were consistent with those of the textbook he used (Thomas' Calculus, 11th edition).

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